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Helmholtz Decomposition.

Helmholtz decomposition theorem

for any vector field $\vec{F}$, twice continuously differentiable (C^2),

$$\vec{F} = -\nabla\Phi + \nabla\times\vec{A}$$

Green function (R^3) : $-\frac{1}{4\pi}\frac{1}{|r-r'|}$

$$F(r) = \int F(r')\,\delta(r-r')\,dV'$$

$$= \int F(r')\left(\frac{-1}{4\pi}\nabla^2\frac{1}{|r-r'|}\right)dV'$$

$$(\nabla\times\nabla\times A)_i = \varepsilon_{ijk}\,\varepsilon_{jmn}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_n}A_m$$

$$= \varepsilon_{kij}\,\varepsilon_{mnj}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_n}A_m$$

$$= (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im})\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_n}A_m$$

$$= \frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_k} - \frac{\partial^2}{\partial x_k^2}A_i$$

$$= (\nabla(\nabla\cdot A) - \nabla^2 A)_i$$

$$= \int F(r')\left(\frac{-1}{4\pi}\right)\left[\nabla\left(\nabla\cdot\frac{1}{|r-r'|}\right) - \nabla\times\nabla\times\frac{1}{|r-r'|}\right]dV'$$

$$= \nabla\int F(r')\left(\frac{-1}{4\pi}\right)\left(-\nabla'\cdot\frac{1}{|r-r'|}\right)dV'$$

$$- \nabla\times\int F(r')\left(\frac{-1}{4\pi}\right)\left(-\nabla'\times\frac{1}{|r-r'|}\right)dV'$$

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$$= \nabla\left[\int \frac{1}{4\pi}\nabla'\cdot\left(\frac{F(r')}{|r-r'|}\right)dv' - \int\frac{1}{4\pi}\frac{\nabla'\cdot F(r')}{|r-r'|}dv'\right]$$

$$-\nabla\times\left[\int\frac{1}{4\pi}\nabla'\times\left(\frac{F(r')}{|r-r'|}\right)dv' - \int\frac{1}{4\pi}\frac{\nabla'\times F(r')}{|r-r'|}dv'\right]$$

$$=\nabla\left[-\int\frac{1}{4\pi}\frac{\nabla'\cdot F(r')}{|r-r'|}dv' + \oint\frac{1}{4\pi}\frac{F(r')}{|r-r'|}\cdot\hat{n}'\,ds'\right]$$

$$-\nabla\times\left[-\int\frac{1}{4\pi}\frac{\nabla'\times F(r')}{|r-r'|}dv' + \oint\frac{1}{4\pi}\hat{n}'\times\frac{F(r')}{|r-r'|}ds'\right]$$

$$= -\nabla\Phi + \nabla\times A$$

$$\Phi = \int\frac{1}{4\pi}\frac{\nabla'\cdot F(r')}{|r-r'|}dv' - \oint\frac{1}{4\pi}\hat{n}'\cdot\frac{F(r')}{|r-r'|}ds'$$

$$A = \int\frac{1}{4\pi}\frac{\nabla'\times F(r')}{|r-r'|}dv' - \oint\frac{1}{4\pi}\hat{n}'\times\frac{F(r')}{|r-r'|}ds'$$

<u>Navier-Stokes equations</u> (incompressible)

$$\rho\frac{du}{dt} = -\nabla p + \mu\nabla^2 u$$

$$u = v + \nabla\phi, \quad \nabla\cdot v = 0, \quad \nabla^2\phi = 0 \quad \therefore \nabla\cdot u = 0$$

$$\therefore \rho\frac{\partial}{\partial t}(v+\nabla\phi) + \rho(v+\nabla\phi)\cdot\nabla(v+\nabla\phi) = -\nabla p + \mu\nabla^2(v+\nabla\phi)$$

$$\rho\frac{\partial v}{\partial t} + \nabla\left(\rho\frac{\partial\phi}{\partial t}\right) + \rho\left[v\cdot\nabla v + v\cdot\nabla\cancel{\phi}(\nabla\phi) + \nabla\phi\cdot\nabla(\overset{v}{\cancel{\nabla\phi}}) + \nabla\phi\cdot\nabla(\nabla\phi)\right]$$

$$+\nabla p = \mu\nabla^2 v + \mu\nabla^2(\nabla\phi)$$

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$$(v\cdot\nabla v)_i = v_j \frac{\partial}{\partial x_j} v_i = \frac{\partial}{\partial x_j}(v_i v_j) - v_i \frac{\partial v_j}{\partial x_j} = (\nabla\cdot(v\otimes v))_i$$

$$(v\cdot\nabla(\nabla\phi))_i = v_j \frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_i}\right) = \frac{\partial}{\partial x_j}\left(v_j \frac{\partial\phi}{\partial x_i}\right) - \frac{\partial v_j}{\partial x_j}\frac{\partial\phi}{\partial x_i} = (\nabla\cdot(v\otimes\nabla\phi))_i$$

$$(\nabla\phi\cdot\nabla v)_i = \frac{\partial\phi}{\partial x_j}\frac{\partial v_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_j} v_i\right) - \frac{\partial^2\phi}{\partial x_j^2} v_i = (\nabla\cdot(\nabla\phi\otimes v))_i$$

$$(\nabla\phi\cdot\nabla(\nabla\phi))_i = \frac{\partial\phi}{\partial x_j}\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_i}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_i}\right) - \frac{\partial^2\phi}{\partial x_j^2}\frac{\partial\phi}{\partial x_i} = \nabla\cdot(\nabla\phi\otimes\nabla\phi)$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_i}\right)\frac{\partial\phi}{\partial x_j} \rightarrow \nabla\phi\cdot\nabla(\nabla\phi) = \frac{1}{2}\nabla(|\nabla\phi|^2)$$

$$\nabla^2(\nabla\phi) = \nabla(\nabla^2\phi) = 0$$

$$\therefore \rho\frac{du}{dt} = -\nabla p + \mu\nabla^2 u$$

$$\downarrow$$

$$\rho\frac{dv}{dt} + \nabla\left[\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}|\nabla\phi|^2 + p\right]$$

$$+\rho\nabla\cdot\left[v\otimes v + v\otimes\nabla\phi + \nabla\phi\otimes v\right] = \mu\nabla^2 v$$